FREEZING TIME OF A PLATE

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The symmetrical problem of the freezing of a plane-parallel plate is solved approximately for boundary conditions of the third kind.

The process in question consists of two stages: 1) attainment of the cryoscopic temperature at the surface and 2) freezing, during which a change in the state of aggregation takes place.

The first stage is familiar [1] and is not considered in this paper. We note only that as a result of the analytic solution of the first-stage problem we accurately know the temperature $t_2(x, 0) = f(x)$, in particular the temperature of the middle plane $t_{mp} = t_2(R, 0)$, which corresponds to the beginning of the second stage ($\tau = 0$), when the temperature t_{cr} is reached at both surfaces (the problem is assumed symmetrical).

If we locate the coordinate origin at the left surface of the plate and denote by ξ the instantaneous position of the freezing boundary, the mathematical formulation of the problem will be as follows [1]:

$$c_1\gamma_1 \frac{\partial t_1}{\partial \tau} = \lambda_1 \frac{\partial^2 t_1}{\partial x^2} \quad (\tau > 0,\ 0 < x < \xi), \tag{1}$$

$$c_2\gamma_2 \frac{\partial t_2}{\partial \tau} = \lambda_2 \frac{\partial^2 t_2}{\partial x^2} \quad (\tau > 0,\ \xi < x < R), \tag{2}$$

$$t_2(x, 0) = f(x), \tag{3}$$

$$\left\{\frac{\partial t_1}{\partial x}\right\}_{x=0} = \frac{\alpha}{\lambda_1}\,[t_1(0, \tau) - t_m], \tag{4}$$

$$\left\{\frac{\partial t_2}{\partial x}\right\}_{x=R} = 0, \tag{5}$$

$$t_1(\xi, \tau) = t_2(\xi, \tau) = t_{cr}, \tag{6}$$

$$\rho\gamma_2 \frac{d\xi}{d\tau} = \lambda_1 \left\{\frac{\partial t_1}{\partial x}\right\}_{x=\xi} - \lambda_2 \left\{\frac{\partial t_2}{\partial x}\right\}_{x=\xi}, \tag{7}$$

where

$$\rho = w\omega L.$$

Equation (7) is Stefan's condition [2], which dates from 1889; it is derived in many monographs (for example [3, 4]).

There is considerable literature concerned with the solution and investigation of problems with a moving boundary, starting with the work of Lamé and Clapeyron in 1831 and ending with the research of L. I. Rubinshtein, which has been published during the last twenty years, including [5]. However, with rare exceptions there has been no success in obtaining an analytic solution of these problems, and the known numerical methods require a considerable expenditure of time and labor.

In applied thermophysics it is necessary to resort to simplified approximate methods giving a solution accurate enough for engineering purposes. Particularly worth noting are those involving the replacement of the true temperature curves with approximate analogs based on various physical considerations. However, even when the temperature curves themselves are closely approximated, Stefan's condition (7) may introduce very large errors; since it also contains the derivatives at the corner point (at $x = \xi$), the slopes of the tangents may differ considerably from the true values.

For this reason the method proposed by L. S. Leibenzon [6] is particularly valuable. This method makes it possible to introduce instead of derivatives determined at the phase interface quantities determined at the surface. For heat conduction problems Leibenzon called his condition the integral condition and noted that his method is analogous to the approximate method represented by Karman's integral condition in boundary layer theory.

Leibenzon's method has been used by many investigators, in particular by A. N. Tikhonov and E. G. Shvidkovskii [7] in their well-known work on the theory of continuous casting.

In connection with the freezing problems that confront the refrigeration industry the approximate formulas of R. Plank (he began this work in 1913 [8]) are well-known. These formulas can be refined by approximately solving systems (1)–(7).

Following Leibenzon's ideas, we will first transform condition (7). Integrating both sides of Eq. (1) from 0 to ξ, we have

$$\lambda_1 \left\{\frac{\partial t_1}{\partial x}\right\}_{x=\xi} - \lambda_1 \left\{\frac{\partial t_1}{\partial x}\right\}_{x=0} = c_1\gamma_1 \int\limits_0^{\xi} \frac{\partial t_1}{\partial \tau}\, dx. \tag{8}$$

Similarly, from (2), integrating from ξ to R, we obtain

$$\lambda_2 \left\{\frac{\partial t_2}{\partial x}\right\}_{x=R} - \lambda_2 \left\{\frac{\partial t_2}{\partial x}\right\}_{x=\xi} = c_2\gamma_2 \int\limits_{\xi}^{R} \frac{\partial t_2}{\partial \tau}\, dx. \tag{9}$$

Finding the quantities $\{\partial t_1/\partial x\}_{x=\xi}$ and $\{\partial t_2/\partial x\}_{x=\xi}$ from (8) and (9) and using the symmetry condition (5), we can rewrite Stefan's condition (7) in the form

$$\rho\gamma_2 \frac{d\xi}{d\tau} = \lambda_1 \left\{\frac{\partial t_1}{\partial x}\right\}_{x=0} + c_1\gamma_1 \int\limits_0^{\xi} \frac{\partial t_1}{\partial \tau}\, dx + c_2\gamma_2 \int\limits_{\xi}^{R} \frac{\partial t_2}{\partial \tau}\, dx. \tag{10}$$

Noting that

$$\frac{\partial t}{\partial \tau} = \frac{\partial t}{\partial \xi}\,\frac{d\xi}{d\tau},$$

we can write condition (10) as follows:

$$\left[\rho\gamma_2 - c_1\gamma_1 \int_0^{\xi} \frac{\partial t_1}{\partial \xi}\,dx - c_2\gamma_2 \int_{\xi}^{R} \frac{\partial t_2}{\partial \xi}\,dx\right]\frac{d\xi}{d\tau} = $$

$$= \lambda_1 \left\{\frac{\partial t_1}{\partial x}\right\}_{x=0}. \quad (11)$$

When the temperature curves $t_1 = t_1(x, \tau)$ and $t_2 = t_2(x, \tau)$ are approximately assigned, it is undoubtedly more accurate to determine the front $\xi = \xi(\tau)$ from the ordinary differential equation with separable variables (11) than from Eq. (7).

We present one possible variant of the approximate solution of problems (1)–(7), taking a linear law of variation of temperature t_1 in the frozen zone (which corresponds to a stationary temperature distribution) and a parabolic law of variation for t_2 in the liquid zone. Then in the equation

$$t_1 = C_1 x + C_2$$

the constants C_1 and C_2 are found from conditions (4) and (6), and therefore

$$t_1(x, \tau) = t_{\mathrm{cr}} + \frac{\alpha}{\lambda_1}\,[t_1(0, \tau) - t_{\mathrm{m}}]\,(x - \xi). \quad (12)$$

In order to eliminate from (12) the temperature at the surface $t_1(0, \tau)$ which is inconvenient for practical computations, we set $x = 0$ and from the equation

$$t_1(0, \tau) = t_{\mathrm{cr}} - \frac{\alpha}{\lambda_1}\,[t_1(0, \tau) - t_{\mathrm{m}}]\,\xi$$

we find

$$t_1(0, \tau) = \left(t_{\mathrm{cr}} + \frac{\alpha\xi}{\lambda_1}\,t_{\mathrm{m}}\right) \Big/ \left(1 + \frac{\alpha\xi}{\lambda_1}\right). \quad (13)$$

Substituting (13) in (12), we finally obtain

$$t_1(x, \tau) = t_{\mathrm{cr}} + \frac{t_{\mathrm{cr}} - t_{\mathrm{m}}}{\lambda_1/\alpha + \xi}\,(x - \xi) \quad (0 \leqslant x \leqslant \xi). \quad (14)$$

It is easy to see that when conditions (5) and (6) are satisfied the parabolic law of variation of t_2 in the interval $\xi \le x \le R$ should be taken in the form

$$t_2(x, \tau) = t_2(R, \tau) + [t_{\mathrm{cr}} - t_2(R, \tau)]\left(\frac{R - x}{R - \xi}\right)^2. \quad (15)$$

The temperature of the middle plane $t_2(R, \tau)$ varies with ξ from the value $t_{\mathrm{mp}} = t_2(R, 0)$ to the value t_{cr} at $\xi = R$; therefore for simplicity we set

$$t_2(R, \tau) = t_{\mathrm{mp}} + (t_{\mathrm{cr}} - t_{\mathrm{mp}})\,\frac{\xi}{R}. \quad (16)$$

Substituting (16) in (15) we finally obtain

$$t_2(x, \tau) = t_{\mathrm{mp}} + \frac{t_{\mathrm{cr}} - t_{\mathrm{mp}}}{R}\left[\xi + \frac{(R - x)^2}{R - \xi}\right]$$

$$(\xi \leqslant x \leqslant R). \quad (17)$$

Using (14) and (15), we evaluate the expressions in Eq. (11):

$$\int_0^{\xi} \frac{\partial t_1}{\partial \xi}\,dx = -\frac{t_{\mathrm{cr}} - t_{\mathrm{m}}}{2}\left[1 - \frac{\lambda_1^2}{\alpha^2(\lambda_1/\alpha + \xi)^2}\right];$$

$$\int_{\xi}^{R} \frac{\partial t_2}{\partial \xi}\,dx = \frac{2(t_{\mathrm{cr}} - t_{\mathrm{mp}})}{3R}\,(R - \xi);$$

$$\left\{\frac{\partial t_1}{\partial x}\right\}_{x=0} = \frac{t_{\mathrm{cr}} - t_{\mathrm{m}}}{\lambda_1/\alpha + \xi}.$$

Substituting these values in (11), we obtain

$$\left[A\left(\frac{\lambda_1}{\alpha} + \xi\right) - \frac{B}{\lambda_1/\alpha + \xi} + C(R - \xi)\left(\frac{\lambda_1}{\alpha} + \xi\right)\right] d\xi = d\tau, \quad (18)$$

where

$$A = \frac{\rho\gamma_2}{\lambda_1(t_{\mathrm{cr}} - t_{\mathrm{m}})} + \frac{c_1\gamma_1}{2\lambda_1};$$

$$B = \frac{c_1\gamma_1\lambda_1}{2\alpha^2}; \quad C = \frac{2c_2\gamma_2(t_{\mathrm{mp}} - t_{\mathrm{cr}})}{3R\lambda_1(t_{\mathrm{cr}} - t_{\mathrm{m}})}.$$

The particular solution of Eq. (18) satisfying the condition $\xi(0) = 0$ has the form

$$\tau = \frac{\xi}{2}\left(\frac{2\lambda_1}{\alpha} + \xi\right)\left[A + C\left(R + \frac{\lambda_1}{\alpha}\right)\right] -$$

$$- B\ln\left(1 + \frac{\alpha\xi}{\lambda_1}\right) - \frac{C}{3}\left[\left(\frac{\lambda_1}{\alpha} + \xi\right)^3 - \frac{\lambda_1^3}{\alpha^3}\right]. \quad (19)$$

In order to determine the freezing time $\tau = \tau_0$ for the entire plate it is necessary to set $\xi = R$ in (19). Thus,

$$\tau_0 = \frac{R}{2\lambda_1}\left(\frac{2\lambda_1}{\alpha} + R\right) \times$$

$$\times \left[\frac{\rho\gamma_2}{t_{\mathrm{cr}} - t_{\mathrm{m}}} + \frac{c_1\gamma_1}{2} + \frac{2c_2\gamma_2(t_{\mathrm{mp}} - t_{\mathrm{cr}})(R + \lambda_1/\alpha)}{3R(t_{\mathrm{cr}} - t_{\mathrm{m}})}\right] -$$

$$- \frac{c_1\gamma_1\lambda_1}{2\alpha^2}\ln\left(1 + \frac{\alpha R}{\lambda_1}\right) -$$

$$- \frac{2c_2\gamma_2(t_{\mathrm{mp}} - t_{\mathrm{cr}})}{3(t_{\mathrm{cr}} - t_{\mathrm{m}})}\left(\frac{\lambda_1}{\alpha^2} + \frac{R}{\alpha} + \frac{R^2}{3\lambda_1}\right). \quad (20)$$

We note that if in condition (10) the terms on the right containing integrals are discarded, the equation assumes the simple form

$$\rho\gamma_2\,\frac{d\xi}{d\tau} = \lambda_1\left\{\frac{\partial t_1}{\partial x}\right\}_{x=0},$$

and then for law (14) finding the total freezing time reduces to integration of the equation

$$d\tau = \frac{\rho\gamma_2}{\lambda_1(t_{\mathrm{cr}} - t_{\mathrm{m}})}\left(\frac{\lambda_1}{\alpha} + \xi\right) d\xi,$$

so

$$\tau_0 = \frac{\rho\gamma_2 R}{t_{cr} - t_m}\left(\frac{R}{2\lambda_1} + \frac{1}{\alpha}\right). \quad (21)$$

Formula (21) is Plank's formula [8]. Thus, formula (20) is an improvement on formula (21).

We note that similar improvements can be introduced in connection with the freezing of spherical and cylindrical bodies by first transforming Stefan's condition (7) to the integral form.

NOTATION

t is the temperature; τ is the time; λ is the thermal conductivity; γ is the specific weight; c is the specific heat; 2R is the thickness of plane-parallel plate; α is the heat transfer coefficient; t_{cr} is the cryoscopic temperature; t_m is the temperature of medium; w is the relative moisture content; ω is the relative amount of frozen-out water; L is the latent heat of melting of ice. The subscript 1 corresponds to the solid phase, the subscript 2 to the liquid phase.

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